Gavin Brown

# KAWAMATA BOUNDEDNESS FOR FANO THREEFOLDS AND THE GRADED RING DATABASE

Miles Reid
Al Kasprzyk
Selma Altınok . . .

| | Genus | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codimension | −2 | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
| 1 | | 54 | 32 | 6 | 2 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1 | 45 | 29 | 6 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | | 26 | 29 | 8 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 3 | 60 | 54 | 15 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | 1 | 58 | 63 | 21 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 4 | 80 | 98 | 35 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | | 81 | 116 | 49 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | 6 | 128 | 182 | 79 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 3 | 107 | 208 | 109 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 10 | 10 | 194 | 314 | 171 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | |
| 11 | 5 | 168 | 378 | 236 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| 12 | 7 | 244 | 538 | 353 | 171 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | |
| 13 | 8 | 254 | 638 | 485 | 236 | 171 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | |
| 14 | 11 | 377 | 879 | 698 | 353 | 236 | 171 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | |
| 15 | 10 | 349 | 889 | 870 | 480 | 351 | 236 | 171 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | |
| 16 | 15 | 509 | 802 | 998 | 675 | 473 | 349 | 235 | 171 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | |
| 17 | 21 | 501 | 480 | 740 | 786 | 637 | 456 | 342 | 233 | 171 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | |
| 18 | 17 | 604 | 284 | 318 | 773 | 658 | 571 | 427 | 331 | 228 | 169 | 109 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | |
| 19 | 23 | 465 | 103 | 43 | 401 | 516 | 488 | 474 | 380 | 308 | 221 | 167 | 108 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | | |
| 20 | 31 | 220 | 44 | 4 | 106 | 159 | 267 | 299 | 345 | 308 | 272 | 204 | 160 | 106 | 79 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | | |
| 21 | 32 | 3 | 5 | | 8 | 24 | 42 | 101 | 137 | 202 | 213 | 213 | 176 | 149 | 101 | 77 | 49 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | | |
| 22 | 27 | | 1 | | 1 | 1 | 3 | 7 | 28 | 43 | 88 | 117 | 141 | 135 | 128 | 94 | 75 | 48 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | | |
| 23 | 13 | | | | | | | 1 | 1 | 5 | 7 | 25 | 43 | 71 | 84 | 98 | 78 | 68 | 46 | 35 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | | |
| 24 | 10 | | | | | | | | | | 1 | 1 | 5 | 10 | 23 | 36 | 59 | 54 | 57 | 41 | 33 | 21 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | | |
| 25 | 4 | | | | | | | | | | | | | 1 | 1 | 4 | 10 | 25 | 27 | 39 | 35 | 31 | 20 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | | |
| 26 | 2 | | | | | | | | | | | | | | | | 1 | 1 | 7 | 9 | 20 | 23 | 24 | 18 | 15 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | | |
| 27 | | | | | | | | | | | | | | | | | | | | 1 | 1 | 6 | 10 | 15 | 13 | 13 | 8 | 6 | 3 | 2 | 1 | 1 | | | | | | | | |
| 28 | | | | | | | | | | | | | | | | | | | | | | | 1 | 2 | 4 | 8 | 11 | 7 | 6 | 3 | 2 | 1 | 1 | | | | | | | |
| 29 | | | | | | | | | | | | | | | | | | | | | | | | | | 1 | 2 | 6 | 5 | 6 | 3 | 2 | 1 | 1 | | | | | | |
| 30 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 2 | 1 | 4 | 3 | 2 | 1 | 1 | | | | | |
| 31 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 2 | 2 | 2 | 1 | 1 | | | | |
| 32 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1 | 2 | 1 | 1 | | | |
| 33 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1 | 1 | | |
| 34 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1 | |
| 35 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1 |

A *Fano 3-fold* is a normal complex 3-dimensional projective variety $X$ with canonical singularities and $-K_X$ ample.

$X \cong \operatorname{Proj} R(X)$

Its *genus* is $g_X := h^0(X, -K_X) - 2$

(total / pluri) *anticanonical ring* $R(X) = \bigoplus_{m \geq 0} H^0(X, -mK_X)$

and *Hilbert series* $P_X = \sum_{m \geq 0} h^0(-mK_X)\, t^m$ = Hilbert series of $R(X)$.

GRDB is an organised list of 39,550 rational functions that may be $P_X$s.

*Thm* $X$ a Fano 3-fold with $-K_X^3 \leq -3K_X c_2$

then $P_X$ is one of the 39,550 rational functions in GRDB.

A **Mori-Fano 3-fold** (or $\mathbb{Q}$-Fano 3-fold) is a Fano 3-fold $X$ with $\mathbb{Q}$-factorial terminal singularities and $\rho_X = 1$.

**Thm** (Kawamata 1992) A Mori-Fano 3-fold $X$ with $\Omega^1_X$ semistable satisfies $-K_X^3 \leq -3K_X c_2$, so $P_X \in$ GRDB.

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**Reminder** Iskovskikh, Mori-Mukai 17 + 88 = 105 families of smooth Fano 3-folds.

17: $\rho_X = 1$

eg $X_4 \subset \mathbb{P}^4$
$X_{2,3} \subset \mathbb{P}^5$

88: $\rho_X \geq 2$

eg Blowup of $X_3 \subset \mathbb{P}^4$ in plane cubic
or $(2,2) \subset \mathbb{P}^2 \times \mathbb{P}^2$.

(See the Fanography webpage by Pieter Belmans.)

Reid, Johnson-Kollár, ...: famous 95 families of weighted hypersurfaces ($-K = \mathcal{O}(1)$)

$(X_4 \subset \mathbb{P}^4)$, $X_5 \subset \mathbb{P}(11112)$, $X_6 \subset \mathbb{P}(11122)$, ..., $X_{66} \subset \mathbb{P}(1,5,6,22,33)$

← cf. Chen-Chen-Chen

Terminal singularities:

$$X_5 : (y^2 x_1 + y A_3 + B_5 = 0) \subset \mathbb{P}(\underset{x_1 \cdots x_4}{1111}\ \underset{y}{2})$$

$A = A(\underline{x})$, $B = B(\underline{x})$ general

$\overset{\cup}{P_y} = (00001)$ as $\frac{1}{2}(111)$ singularity. $\left(\cong 0 \in \mathbb{C}^3/(\mathbb{Z}/_2)\ (\varepsilon x, \varepsilon y, \varepsilon z)\right)$

$X_6 \subset \mathbb{P}(11122)$ meets the index 2 $\mathbb{P}(22)$ axis at 3 points in general:

cf. Sano ↙

"basket" of singularities $\mathcal{B}_X = \{3 \times \frac{1}{2}(111)\}$

⋮

$X_{66} \subset \mathbb{P}(1,5,6,22,33)$ has $\mathcal{B}_X = \{\frac{1}{2}(111), \frac{1}{3}(112), \frac{1}{5}(123), \frac{1}{11}\{1,5,6\}\}$

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With Al Kasprzyk: 11,618 weighted hypersurface Mori-Fano 4-folds.

Thm (RR, Kawamata 86, Reid, Barlow) $X$ a Fano 3-fold with sings $B = \{\frac{1}{r}(1,a,-a)\}$

$ab \equiv 1 \mod r$

Thm (1) $-K_X^3 = 2g-2 + \sum_B \frac{b(r-b)}{r}$

(2) $24 = -K_X c_2 + \sum_B \left(r - \frac{1}{r}\right)$.

Simple calc: 8314 baskets $B$ have $\sum_B r - \frac{1}{r} < 24$.

Then for each basket $0 < -K_X^3 \leq -3K_X c_2$

with (1) + (2) bounds $g$ $\longrightarrow$ 39,550 $(B, g)$ pairs.

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Fletcher-Reid plurigenus formula (= RR):

$$P_X(t) = \frac{1+t}{(1-t)^2} - \frac{t(1+t)}{(1-t)^4}\frac{K_X^3}{2} - \sum_B (\text{something})$$

i.e. $(B_X, g) \leftrightarrow P_X$ are equivalent data.

# Toric Fano 3-folds

[Kasprzyk 10] classifies these (with canonical singularities).

→ 674,688 lattice polytopes. (listed separately on GRDB)

These provide examples for 5610 of the 39550 Hilbert series.

(12 do not lie among 39,550 — but in larger list 52,646 : $-K^3 \leq -4Kc_2$

eg $\mathbb{P}(1135)$.)

# Prokhorov champions, Cheltsov, Karzhemanov.

**Thm** (Prokhorov 05) $g = 37$, $-K_X^3 = 72$ are maximal for Fano 3-folds with canonical Gorenstein singularities.

Moreover, in this case $X \cong \mathbb{P}(1113)$ or $\mathbb{P}(1146)$.

**Thm** (Karzhemanov 09)

$g = 36$ : $X \cong Bl_p\, \mathbb{P}(1146)$ ~ index 2 point

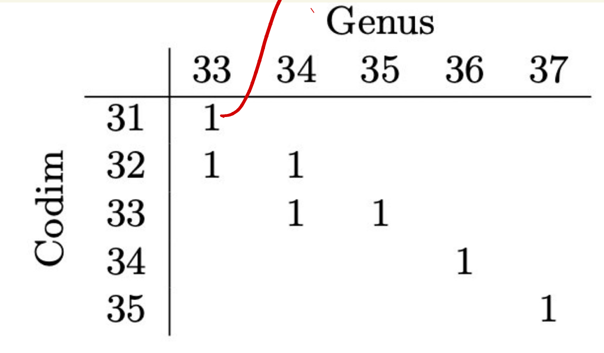
$g = 35$ : does not occur.

$g = 34$ : $\Phi_{-K}(\mathcal{U})$ for $\mathcal{U} = \mathrm{Proj}_{\mathbb{P}^1}\, \mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(5)$

$\downarrow$

[birational to $\mathbb{P}(1135)$]

$\mathbb{P}^3$



| Codim | Genus 33 | 34 | 35 | 36 | 37 |
|---|---|---|---|---|---|
| 31 | 1 | | | | |
| 32 | 1 | 1 | | | |
| 33 | | 1 | 1 | | |
| 34 | | | | 1 | |
| 35 | | | | | 1 |

## Codimension / embedding estimates. : given $P$, what might $X$ look like for $P_X = P$ ?

Eg $\quad P_X = \dfrac{1-t^6}{(1-t)^3(1-t^2)(1-t^3)}$ when $X = X_6 \subset \mathbb{P}(1\,1\,1\,2\,3)$

When suitably expressed, $P_X$ encodes ambient weights in denominator and betti data of free resolution of $\mathcal{O}_X$ in numerator.

This idea often gives good candidates in low codimension ($\leq 3$ ish) which we can test. In higher codim, often misleading.

# Projection / unprojection

$-K_Y$ nef & big

$$\begin{tikzcd}
& \mathbb{P}^2 \cong E \subset Y \arrow[dl, "\text{Kawamata blowup}"'] \arrow[dr, "\text{contracts 16 flopping } \mathbb{P}^1\text{'s}"] & \\
X_{4,4} \subset \mathbb{P}(111222) & & \mathbb{P}^2 \cong D \subset Z_6 \subset \mathbb{P}(111\,2\,2)
\end{tikzcd}$$

$4\times\frac{1}{2}(111)$

$X_{4,4} \subset \mathbb{P}(111222)$

$x_1 \cdots\ y_1 \cdots y_3$

#11435

$3\times\frac{1}{2}(111)$

$Z_6 \subset \mathbb{P}(111\,2\,2)$

$x \cdots\ y_1\ y_2$

#11101

$$\begin{cases} y_3 y_1 = f_4(\underline{x}, y_1, y_2) \\ y_3 y_2 = g_4(\underline{x}, y_1, y_2) \end{cases} \quad \dashrightarrow \quad (y_2 f = y_1 g)$$

eliminate $y_3$

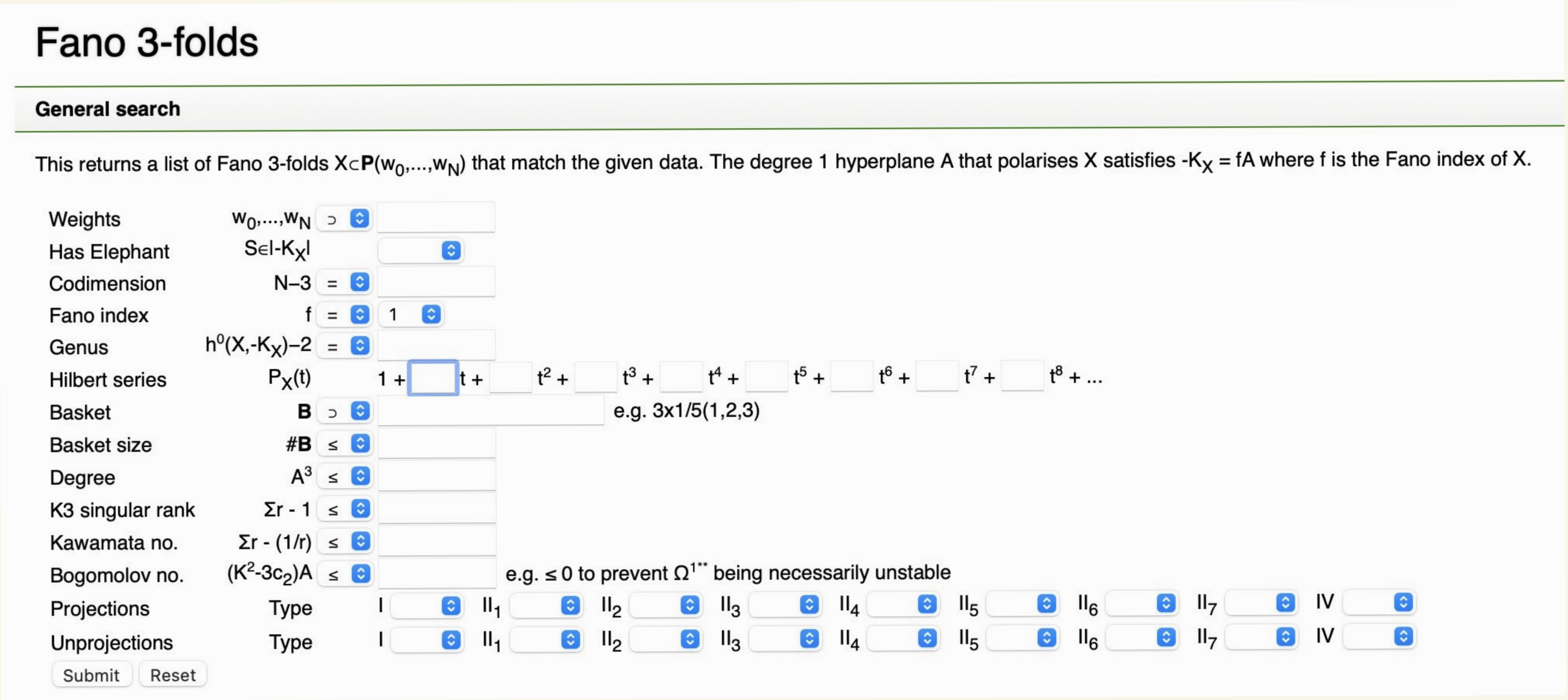
$$\cup$$

$$(y_1 = y_2 = 0) = \mathbb{P}^2$$

$f, g$ general

! singular at $y_1 = y_2 = f = g = 0$

$= 16$ ODPs on $D$.



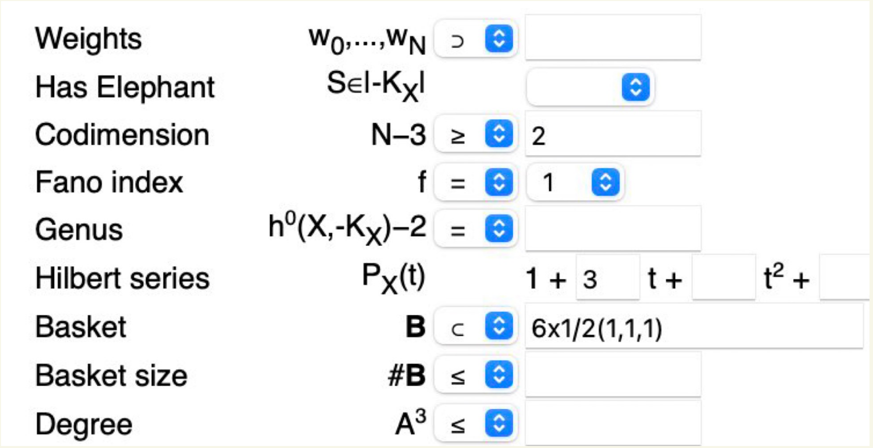
# Fano 3-folds

## General search

This returns a list of Fano 3-folds $X\subset\mathbf{P}(w_0,\dots,w_N)$ that match the given data. The degree 1 hyperplane A that polarises X satisfies $-K_X = fA$ where f is the Fano index of X.

| | | | |
|---|---|---|---|
| Weights | $w_0,\dots,w_N$ | ⊃ | |
| Has Elephant | $S\in\lvert -K_X\rvert$ | | |
| Codimension | N–3 | = | |
| Fano index | f | = | 1 |
| Genus | $h^0(X,-K_X)-2$ | = | |
| Hilbert series | $P_X(t)$ | | $1 + \quad t + \quad t^2 + \quad t^3 + \quad t^4 + \quad t^5 + \quad t^6 + \quad t^7 + \quad t^8 + \dots$ |
| Basket | **B** | ⊃ | e.g. 3x1/5(1,2,3) |
| Basket size | #**B** | ≤ | |
| Degree | $A^3$ | ≤ | |
| K3 singular rank | Σr - 1 | ≤ | |
| Kawamata no. | Σr - (1/r) | ≤ | |
| Bogomolov no. | $(K^2-3c_2)A$ | ≤ | e.g. ≤ 0 to prevent $\Omega^{1**}$ being necessarily unstable |
| Projections | Type | | I, $II_1$, $II_2$, $II_3$, $II_4$, $II_5$, $II_6$, $II_7$, IV |
| Unprojections | Type | | I, $II_1$, $II_2$, $II_3$, $II_4$, $II_5$, $II_6$, $II_7$, IV |

Eg. query



**Weights** $w_0,\dots,w_N$ ⊃

**Has Elephant** $S\in|-K_X|$

**Codimension** $N-3$ ≥ 2

**Fano index** $f$ = 1

**Genus** $h^0(X,-K_X)-2$ =

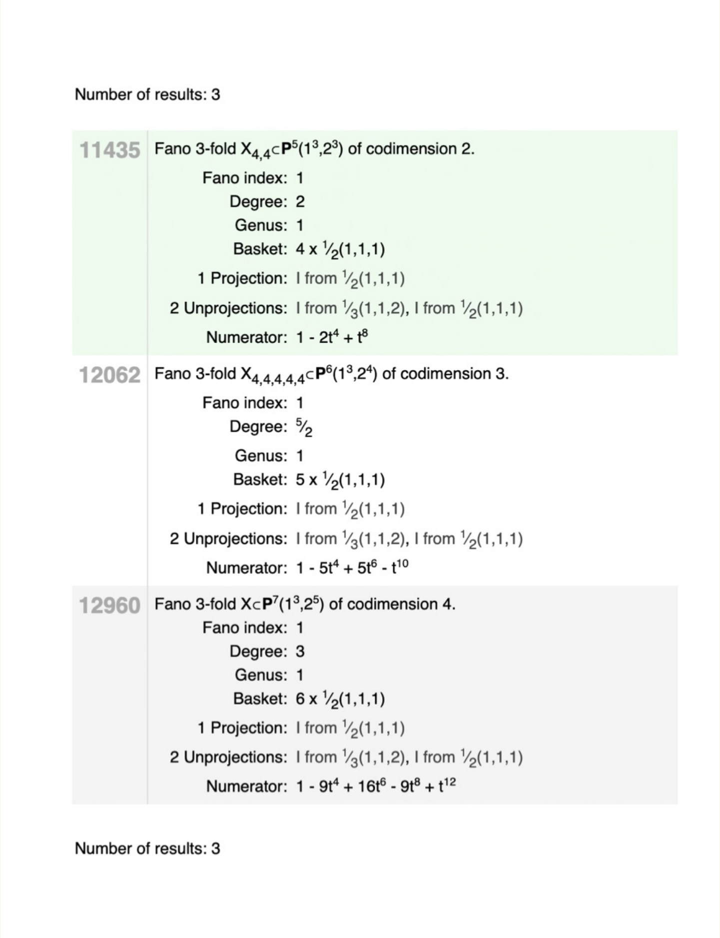
**Hilbert series** $P_X(t)$ $1 + 3\,t + \_\,t^2 + \_$

**Basket** **B** ⊂ 6x1/2(1,1,1)

**Basket size** **#B** ≤

**Degree** $A^3$ ≤

gives results



Number of results: 3

**11435** Fano 3-fold $X_{4,4} \subset \mathbf{P}^5(1^3,2^3)$ of codimension 2.

Fano index: 1
Degree: 2
Genus: 1
Basket: $4 \times \frac{1}{2}(1,1,1)$
1 Projection: I from $\frac{1}{2}(1,1,1)$
2 Unprojections: I from $\frac{1}{3}(1,1,2)$, I from $\frac{1}{2}(1,1,1)$
Numerator: $1 - 2t^4 + t^8$

**12062** Fano 3-fold $X_{4,4,4,4,4} \subset \mathbf{P}^6(1^3,2^4)$ of codimension 3.

Fano index: 1
Degree: $\frac{5}{2}$
Genus: 1
Basket: $5 \times \frac{1}{2}(1,1,1)$
1 Projection: I from $\frac{1}{2}(1,1,1)$
2 Unprojections: I from $\frac{1}{3}(1,1,2)$, I from $\frac{1}{2}(1,1,1)$
Numerator: $1 - 5t^4 + 5t^6 - t^{10}$

**12960** Fano 3-fold $X \subset \mathbf{P}^7(1^3,2^5)$ of codimension 4.

Fano index: 1
Degree: 3
Genus: 1
Basket: $6 \times \frac{1}{2}(1,1,1)$
1 Projection: I from $\frac{1}{2}(1,1,1)$
2 Unprojections: I from $\frac{1}{3}(1,1,2)$, I from $\frac{1}{2}(1,1,1)$
Numerator: $1 - 9t^4 + 16t^6 - 9t^8 + t^{12}$

Number of results: 3

More generally, if similar diagram holds starting from the Kawamata blowup of $\frac{1}{r}(1,a,-a) \in X$ then

$$B_Z = B_X \setminus \left\{ \tfrac{1}{r}(1,a,-a) \right\} \cup \left\{ \tfrac{1}{a}(1,r,-r), \tfrac{1}{r-a}(1,r,-r) \right\}.$$

Conversely, if you can construct $Z \supset D = \mathbb{P}(1,a,r-a)$

Then you might UNPROJECT it to recover $X$.

Kustin–Miller 83

Papadakis–Reid 00

The GRDB presents a model $X \subset w\mathbb{P}$ for each series $P$ that is compatible with all such projections.

# Care is required: degenerations and more.

#20521 $P = \dfrac{1-t^4}{(1-t)^5}$ is recorded in GRDB as $X_4 \subset \mathbb{P}^4$.

The double cover of $Y_2 \subset \mathbb{P}^4$ lies on the boundary, as $X_{2,4} \subset \mathbb{P}(1^5 2)$

↗ Not recorded in GRDB.

(cf. Fletcher calculates 85 families in codim 2 — [CCC] confirms)

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That's fine, but: unproj of degen is not nec degen of unproj:

#548 $X \subset \mathbb{P}(1,3,4,5,6,7,10)$ has another family $Y \subset \mathbb{P}(1\,3\,4\,5\,6\,7\,\underline{\underline{9}}\,10)$

(codim 3 Pfaffians) (codim 4)

! Same Hilbert series $P_X = P_Y$, different codimension (general) models.

$(e_X \neq e_Y)$

# Formats

Mukai: some classical (smooth) Fanos can be modelled on well-known "key" varieties such as Grassmannians.

Eg degree 14 Fano $X \subset \mathbb{P}^9$ (codimension 6)

$$\begin{array}{l} X \\ \| \\ \underset{\text{Plücker}\ \cap\ \mathbb{P}^{14}}{\mathrm{Gr}(2,6)} \cap H_1 \cap \cdots \cap H_5 \end{array}$$

ie its equations are

$$\mathrm{Pf}_4 \begin{pmatrix} \text{skew } 6\times 6 \text{ matrix} \\ \text{of linear forms} \\ \text{on } \mathbb{P}^9 \end{pmatrix} = \underline{0}$$

Grojnowski-Corti-Reid: weighted Grassmannians.

eg general graded skew 5×5 matrix of degrees

$$M = \begin{pmatrix} \times & 2 & 2 & 2 & 2 \\ & \times & 2 & 2 & 2 \\ & & \times & 2 & 2 \\ & & & \times & 2 \\ & & & & \times \end{pmatrix}$$

in variables of $\mathbb{P}^6(111\,2222) = w\mathbb{P}$

then $(\mathrm{Pf}_4 M = 0) \subset w\mathbb{P}$ is a Fano with

$$\mathcal{B} = \{5 \times \tfrac{1}{2}(111)\}$$

$$P_X = \frac{1 - 5t^4 + 5t^6 - t^{10}}{(1-t)^3(1-t^2)^4}.$$

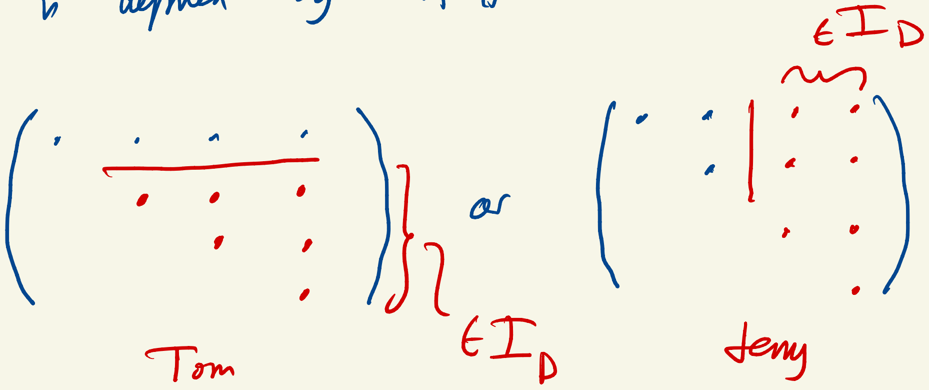
→ 69 similar constructions in codim 3.

# Unprojection : Tom & Jerry — unprojection from codim 3 Pfaffians.

We don't have effective formats in codim $\geq 4$, so unproject from codim 3.

**Point:** How to impose a divisor $D \subset X \subset w\mathbb{P}^6$ if $X$ is defined by Pfaffians.

Idea: (Schubert cycles) either



Sometimes a general matrix of this form determines nodal $X \supset D$ $\rightsquigarrow$ which unprojects to Fano $Y \subset w\mathbb{P}^7$ in codim 4.

→ several hundred families.

## Fanosearch

Corti, Coates, Galkin, Golyshev, Kasprzyk, ...

Aim to classify mirrors of Fanos.

Eg the 1st toric polytope P in Kasprzyk's list has the Hilbert series of some $X \subset \mathbb{P}^{10}(1^7, 2^4)$ with $B_X = 4 \times \frac{1}{2}(111)$ — but with 6 additional singularities that each have 2 smoothing components ⇝ potential for several families

Fanosearch finds 11 maximally mutable Laurent polys supported on P

Conjecturally ⇝ 11 families of Fanos mirror to these.